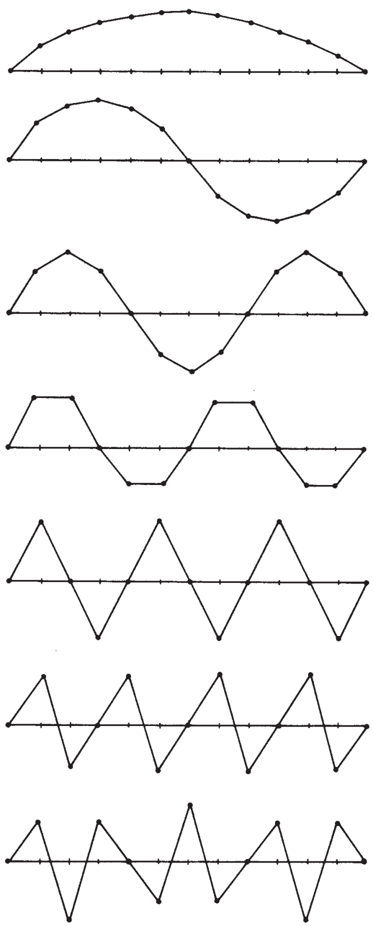

$$k = 1 : \mathbf{w}_{1,j} = \sin\left(\frac{j\pi}{12}\right) \qquad l = 2 = 24h$$

$$k = 2 : \mathbf{w}_{2,j} = \sin\left(\frac{j\pi}{6}\right) \qquad l = 1 = 12h$$

$$k = 3 : \mathbf{w}_{3,j} = \sin\left(\frac{j\pi}{4}\right) \qquad l = \frac{2}{3} = 8h$$

$$k = 4 : \mathbf{w}_{4,j} = \sin\left(\frac{j\pi}{3}\right) \qquad l = \frac{1}{2} = 6h$$

$$k = 6 : \mathbf{w}_{6,j} = \sin\left(\frac{j\pi}{2}\right) \qquad l = \frac{1}{3} = 4h$$

$$k = 8 : \mathbf{w}_{8,j} = \sin\left(\frac{2j\pi}{3}\right) \qquad l = \frac{1}{4} = 3h$$

$$k = 9 : \mathbf{w}_{9,j} = \sin\left(\frac{3j\pi}{4}\right) \qquad l = \frac{2}{9} = \frac{8}{3}h$$

Figure 2.6: *Graphs of the Fourier modes of A on a grid with $n = 12$ points. Modes with wavenumbers $k = 1, 2, 3, 4, 6, 8, 9$ are shown. The wavelength of the kth mode is $\ell = \frac{24h}{k}$.*

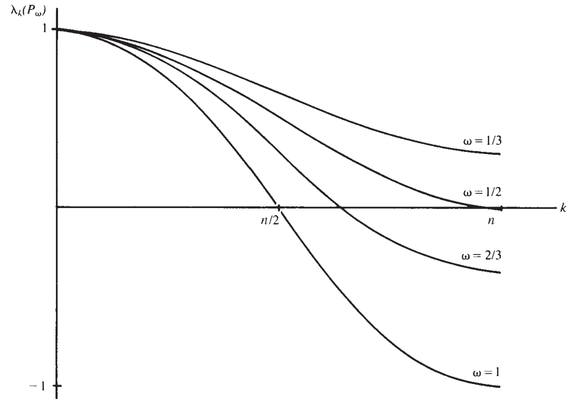

Figure 2.7: *Eigenvalues of the iteration matrix R_ω for $\omega = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$. The eigenvalues $\lambda_k = 1 - 2\omega \sin^2\left(\frac{k\pi}{2n}\right)$ are plotted as if k were a continuous variable on the interval $0 \le k \le n$. In fact, $1 \le k \le n-1$ takes only integer values.*

Recall that for $0 < \omega \le 1$, we have $|\lambda_k(R_\omega)| < 1$. We would like to find the value of ω that makes $|\lambda_k(R_\omega)|$ as small as possible for all $1 \le k \le n-1$. Figure 2.7 is a plot of the eigenvalues λ_k for four different values of ω. Notice that for all values of ω satisfying $0 < \omega \le 1$,

$$\lambda_1 = 1 - 2\omega \sin^2\left(\frac{\pi}{2n}\right) = 1 - 2\omega \sin^2\left(\frac{\pi h}{2}\right) \approx 1 - \frac{\omega \pi^2 h^2}{2}.$$

This fact implies that λ_1, the eigenvalue associated with the smoothest mode, will always be close to 1. Therefore, no value of ω will reduce the smooth components of the error effectively. Furthermore, the smaller the grid spacing h, the closer λ_1 is to 1. Any attempt to improve the accuracy of the solution (by decreasing the grid spacing) will only worsen the convergence of the smooth components of the error. Most basic relaxation schemes share this ironic limitation.

Having accepted the fact that no value of ω damps the smooth components satisfactorily, we ask what value of ω provides the best damping of the oscillatory components (those with $\frac{n}{2} \le k \le n-1$). We could impose this condition by requiring that

$$\lambda_{n/2}(R_\omega) = -\lambda_n(R_\omega).$$ ≦

Solving this equation for ω leads to the optimal value $\omega = \frac{2}{3}$.

We also find (Exercise 13) that with $\omega = \frac{2}{3}$, $|\lambda_k| < \frac{1}{3}$ for all $\frac{n}{2} \le k \le n-1$. This says that the oscillatory components are reduced at least by a factor of three with each relaxation. This damping factor for the oscillatory modes is an important

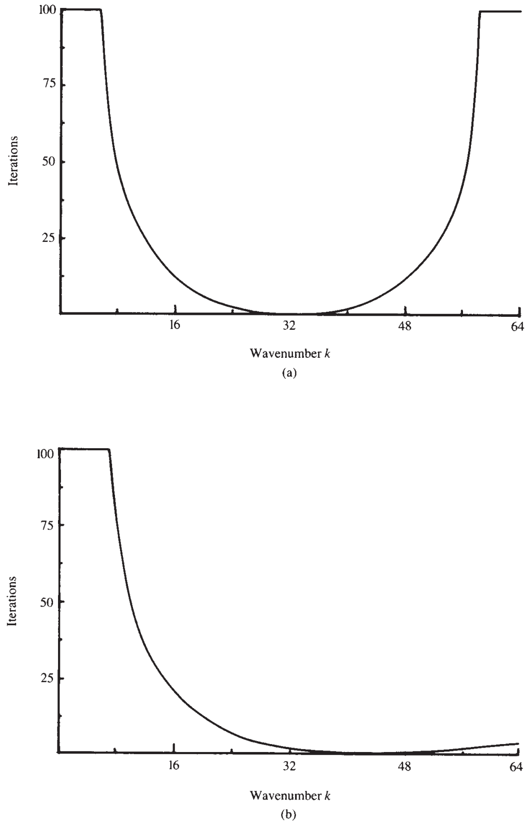

Figure 2.8: *Weighted Jacobi method with* (a) $\omega = 1$ *and* (b) $\omega = \frac{2}{3}$ *applied to the one-dimensional model problem with* $n = 64$ *points. The initial guesses consist of the modes* $\mathbf{w}_k$ *for* $1 \le k \le 63$. *The graphs show the number of iterations required to reduce the norm of the initial error by a factor of* 100 *for each* $\mathbf{w}_k$. *Note that for* $\omega = \frac{2}{3}$, *the damping is strongest for the oscillatory modes* ($32 \le k \le 63$).

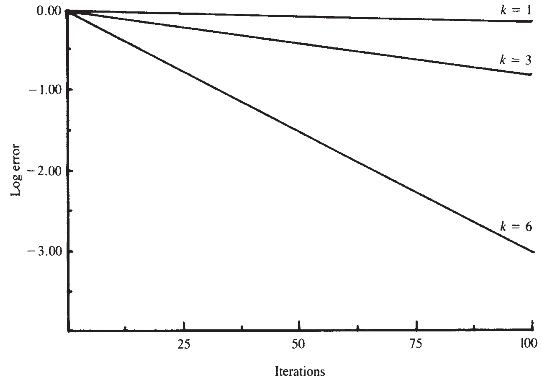

Figure 2.4: *Weighted Jacobi iteration with* $\omega = \frac{2}{3}$ *applied to the one-dimensional model problem with* $n = 64$ *points and with initial guesses* $\mathbf{v}_1$, $\mathbf{v}_3$, *and* $\mathbf{v}_6$. *The log of* $\|\mathbf{e}\|_\infty$ *is plotted against the iteration number for* 100 *iterations.*

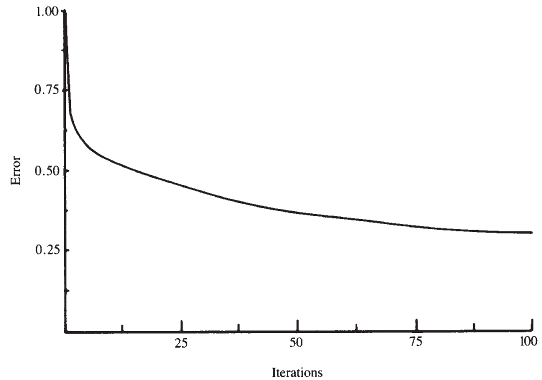

Figure 2.5: *Weighted Jacobi method with* $\omega = \frac{2}{3}$ *applied to the one-dimensional model problem with* $n = 64$ *points and an initial guess* $(\mathbf{v}_1 + \mathbf{v}_6 + \mathbf{v}_{32})/3$. *The maximum norm of the error,* $\|\mathbf{e}\|_\infty$, *is plotted against the iteration number for* 100 *iterations.*