Distribution of the algebraic and discretization error in numerical solution of 1D Poisson model problem

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Outline

Introduction

Real-world problem (e.g. steady-state distribution of temperature)

$\downarrow$ modeling

Mathematical model (Poisson equation $-\Delta u = f \quad + \text{BC}$)

$\downarrow$ discretization

Discretized problem ($a(u_h, v_h) = (f, v_h) \; \forall v_h \quad \Leftrightarrow \quad Ax = b$)

$\downarrow$ algebraic solution

Approximation to the solution(s) ($\tilde{u}_h \neq u_h$)

Introduction

Real-world problem (e.g. steady-state distribution of temperature)

$\downarrow$ modeling

$\downarrow$ error of the model

Mathematical model (Poisson equation $-\Delta u = f \quad + \text{BC}$)

$\downarrow$ discretization

$\downarrow$ discretization error

Discretized problem ($a(u_h, v_h) = (f, v_h)\ \forall v_h \quad \Leftrightarrow \quad Ax = b$)

$\downarrow$ algebraic solution

$\downarrow$ algebraic error (truncation + rounding)

Approximation to the solution(s) ($\tilde{u}_h \neq u_h$)

Introduction

Proposition:

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- What does the balancing of the errors mean?
- How to measure and compare the errors?

FEM discretization

(global) problem

$\downarrow$ local basis function

sparse system matrix

$\downarrow$ algebraic solver

coefficients for global approximation

Model problem

1D Poisson

$$\begin{aligned} -u''(x) &= -12x^2 + 12x + 2\,, \qquad x \in (0,1) \\ u(0) &= u(1) = 0 \end{aligned}$$

with the solution

$$u(x) = (x-2)\,(x-1)\,x\,(x+1)$$

Discretization

using the piecewise linear basis functions on uniform mesh with 19 inner nodes (i.e. with the mesh size $h = 1/20$)

Model problem

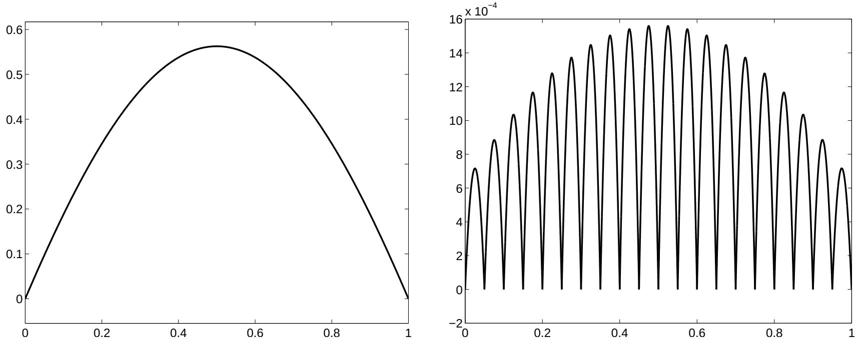

Figure: *The exact solution u (left) and the discretization error $u - u_h$ (right); the right vertical axis is scaled by 10^{-4}.*

Model problem

Discrete solution u_h

- determined by the exact solution x of the stiffness system (x ... coefficients with respect to the basis)
- minimizes the energy norm $\|(u - v_h)'\|$ over the space of piecewise linear functions $\quad \rightarrow \quad$ energy norm is descriptive

Approximation to discrete solution

let $x_n \approx x$, we denote by $u_h^{(n)}$ the approximation to u_h determined by the vector x_n

Model problem - errors

as a simple consequence of the Galerkin orthogonality

$$\begin{aligned}\|(u - u_h^{(n)})'\|^2 &= \|(u - u_h)'\|^2 + \|(u_h - u_h^{(n)})'\|^2 \\ &= \|(u - u_h)'\|^2 + \|x - x_n\|_A^2 \,.\end{aligned}$$

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Discretization error
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Algebraic error (at 9th iteration of CG with $x_0 = 0$)
$\|x - x_9\|_A^2 = 2.5563 \times 10^{-4}$, normwise backward error $= 8.5921 \times 10^{-4}$

Total error
$\|(u - u_h^{(9)})'\|^2 = 3.7556 \times 10^{-3}$

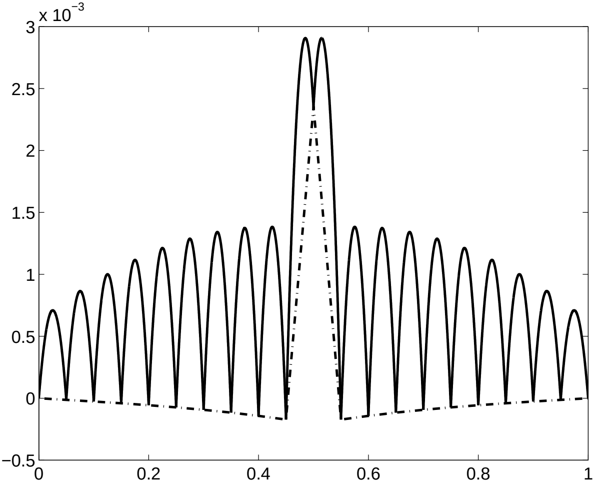

Figure: *The algebraic error* $u_h - u_h^{(9)}$ *(dashed-dotted) and the total error* $u - u_h^{(9)}$ *(solid) after 9 CG iteration steps; the vertical axis is scaled by* 10^{-3}.

Comments

Localization of the algebraic error is not the artefact of particular setup, we observed it for

- different right hand sides,
- various number of discretization nodes,
- various number of CG iteration steps,
- nonuniform meshes.

Interpretation of the algebraic error

Interpret the inaccuracies in the solution process in terms of meaningful modification of the mathematical model

Possibilities:

- perturbation of the weak formulation, *functional backward error* [Arioli et al. '01]
- change of the FEM basis, *energy backward error* [Gratton, Jiránek, Vasseur '12], leads to the loss of locallity
- change of the discretization mesh

Interpretation of the alg. error via the change of the discretization mesh

Aim:

find a new mesh such that the discretization error on the mesh "resembles" the total error in the model problem.

No claim for optimality or generality, just for illustration.

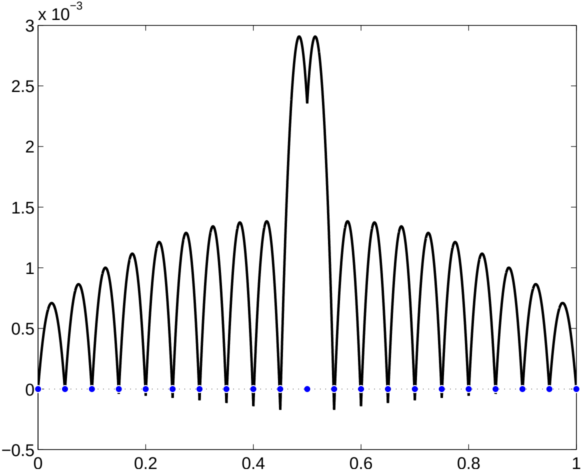

Figure: *The total error (solid) after 9 CG iteration steps with inner discretization nodes (blue points).*

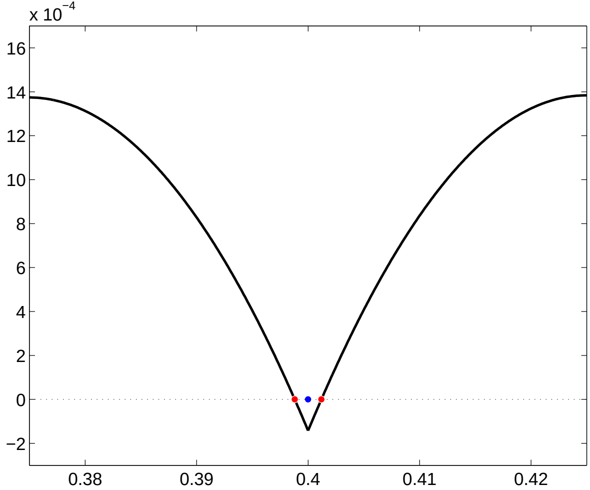

Figure: *Replacing the inner node by two roots of the total error.*

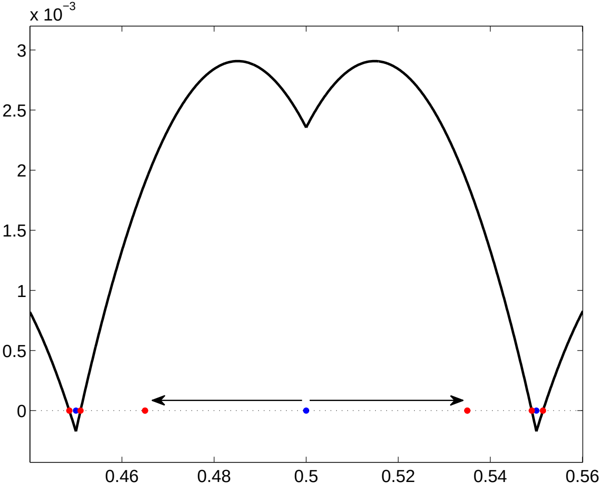

Figure: *Replacing the central node by two nodes* $0.5 \pm 0.7h$.

Change of the discretization mesh

- 18 inner nodes are replaced by 18 closed couples
- central node 0.5 is replaced by two nodes defined as $0.5 \pm 0.7h$, the length of the new central element is $1.4h$

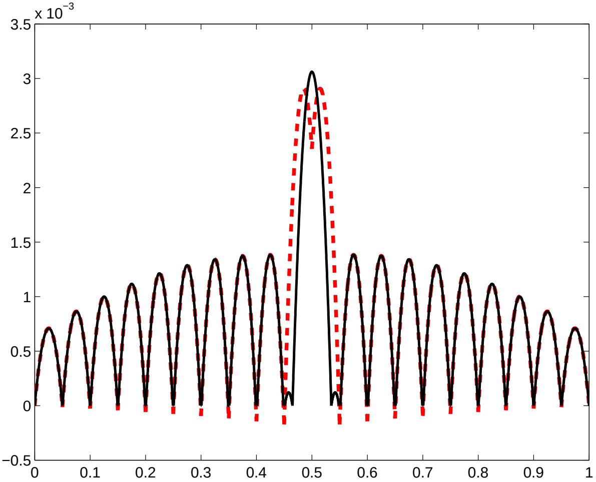

Figure: The total error for the original mesh (red dashed line) and the discretization error on the modified mesh (solid line); the vertical axis is scaled by 10^{-3}.

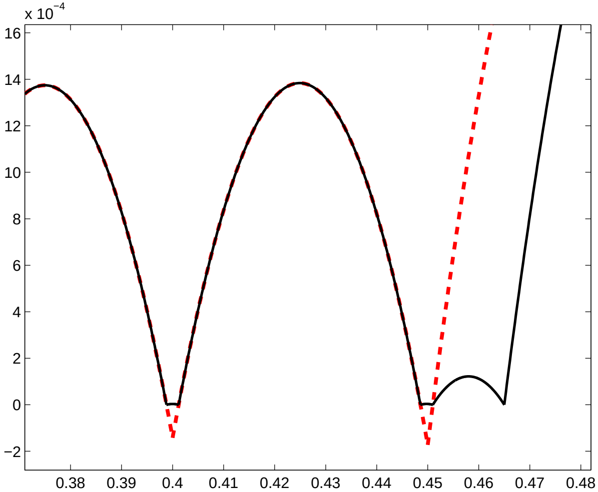

Figure: The detail; the vertical axis is scaled by 10^{-4}.

Change of the discr. mesh - comments

The example demonstrates that the approximated solution which includes the algebraic error can be interpreted (here with a small inaccuracy) as the discrete solution on a mesh that can have "holes" in the area of interest.

Conclusion

We showed important phenomena which should be taken into account when solving (large scale) mathematical modeling problems in general.

In particular, for the FEM discretization:

- distribution of the algebraic error may signifficantly differ from the distribution of the discretization error,
- global values (norms) give no information about the local behavior of errors,
- stopping criteria for algebraic solvers should balance the discretization and algebraic parts of the error locally.

Acknowledgement and references

Acknowledgement

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References

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On distribution of the discretization and algebraic error in 1D Poisson model problem,
in preparation.

The end

Thank you for your attention!